types, which in general contain both old and young stars, will be a mixture of stars that have quite different kinematic properties. Indeed, within almost any group of stars chosen according to spectral type only, there will be a certain degree of kinematic heterogeneity, either because the group contains stars of widely differing ages or because it is composed of subgroups formed under rather different initial conditions. We shall return to these points in Chapter 7.

Basic Equations

We now derive the basic equations required for the analysis. Let d be the distance from the Sun to the star, and, because the observational data for proper motions are always given in the equatorial system, let us resolve positions and velocities into equatorial rectangular components. Then (see Figure 6-5),

$$x = d \cos \delta \cos \alpha \tag{6-9a}$$

$$y = d \cos \delta \sin \alpha \tag{6-9b}$$

$$z = d \sin \delta \tag{6-9c}$$

Differentiating equations (6-9) with respect to time, we have the equatorial components of the stellar velocity with respect to the Sun.

$$\dot{x} = X_* - X_\odot = \dot{d} \cos \delta \cos \alpha - \dot{\delta} d \sin \delta \cos \alpha - \dot{\alpha} d \cos \delta \sin \alpha \tag{6-10a}$$

$$\dot{y} = Y_* - Y_\odot = \dot{d} \cos \delta \sin \alpha - \dot{\delta} d \sin \delta \sin \alpha + \dot{\alpha} d \cos \delta \cos \alpha \tag{6-10b}$$

$$\dot{z} = Z_* - Z_\odot = \dot{d} \sin \delta + \dot{\delta} d \cos \delta \tag{6-10c}$$

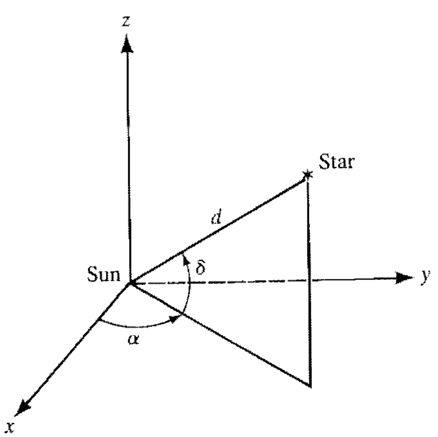

Figure 6-5. Equatorial rectangular coordinate system.

where (X_*, Y_*, Z_*) and $(X_\odot, Y_\odot, Z_\odot)$ are the velocity components of the star and the Sun, respectively, along the (x, y, z) axes in the equatorial system. The quantity $\dot{d}$ stands for the radial velocity, and $\dot{\delta} d$ and $\dot{\alpha} d \cos \delta$ are transverse velocities. These will be expressed in kilometers per second if $\dot{\alpha}$ and $\dot{\delta}$ are expressed in radians per second and d is expressed in kilometers. (We shall introduce more practical units shortly.)

To facilitate the analysis, we must solve equations (6-10) for $\dot{d}$, $\dot{\alpha} d \cos \delta$, and $\dot{\delta} d$, so that we can write equations that involve only radial velocities or only proper motions. The solution is easy to write because the matrix of coefficients in equations (6-10) is orthogonal, so that its inverse equals its transpose. We thus find

$$\dot{x} \cos \delta \cos \alpha + \dot{y} \cos \delta \sin \alpha + \dot{z} \sin \delta = \dot{d} \tag{6-11a}$$

$$-\dot{x} \sin \delta \cos \alpha - \dot{y} \sin \delta \sin \alpha + \dot{z} \cos \delta = \dot{\delta} d \tag{6-11b}$$

$$-\dot{x} \sin \alpha + \dot{y} \cos \alpha = \dot{\alpha} d \cos \delta \tag{6-11c}$$

Solution from Radial Velocities

Let us now examine how equation (6-11a) can be used to derive the solar motion from radial velocity data alone. Suppose we have chosen N stars uniformly distributed over the sky. For each star, we can write an equation of the form of equation (6-11a). Thus, for the ith star,

$$\dot{x}_i \cos \delta_i \cos \alpha_i + \dot{y}_i \cos \delta_i \sin \alpha_i + \dot{z}_i \sin \delta_i = \dot{d}_i \, (i = 1, \ldots, N) \tag{6-12}$$

By definition, the solar motion is determined relative to the centroid of observed stellar velocities. Therefore, we demand that the observed stellar space velocities average to zero, that is, $\langle X_* \rangle = \langle Y_* \rangle = \langle Z_* \rangle = 0$, where $\langle X_* \rangle \equiv (\sum_{i=1}^{N} X_{*i})/N$. Now, even though the coefficients $\cos \delta_i \cos \alpha_i$, $\cos \delta_i \sin \alpha_i$, and $\sin \delta_i$ are, strictly speaking, different from star to star, we can imagine that the sky is divided into small regions within which these coefficients are practically constant. Then, provided that we have chosen a large enough sample to assure that each region contains several stars, it is clear that the averages $\langle X_{*i} \rangle$ and so on can be set to zero region by region. That is, with the assumptions of a zero centroid velocity and a statistically satisfactory sample, we can, in effect, simply drop all terms involving X_{*i}, Y_{*i}, and Z_{*i} out of the left-hand side of equation (6-12). Equations (6-12) are thus effectively of the form

$$a_i X_\odot + b_i Y_\odot + c_i Z_\odot = k_i \qquad (i = 1, \ldots, N) \tag{6-13}$$

The system of equations (6-13) contains N equations in three unknowns. The system is therefore *overdetermined.* We can make an optimum solution for $(X_\odot, Y_\odot, Z_\odot)$ by the *method of least squares*. From the outset, we recognize that it is not possible to make a single choice for $(X_\odot, Y_\odot, Z_\odot)$ that will solve every one of equations (6-13) exactly. (This is true even for perfect data because the stellar velocity components can be set to zero only on the average.) For any specific choice of the solution $(X_\odot, Y_\odot, Z_\odot)$, each equation will yield a *residual* r_i,

$$r_i \equiv k_i - a_i X_\odot - b_i Y_\odot - c_i Z_\odot \qquad (i = 1, \ldots, N) \quad (6\text{-}14)$$

Thus we choose the particular solution that minimizes the sum of the squares of the residuals $R \equiv \sum_{i=1}^{N} r_i^2$. To do this, we demand that $(\partial R/\partial X_\odot) = (\partial R/\partial Y_\odot) = (\partial R/\partial Z_\odot) = 0$. It is easy to show that these requirements produce an equation of the form

$$X_\odot \sum_{i=1}^{N} a_i^2 + Y_\odot \sum_{i=1}^{N} a_i b_i + Z_\odot \sum_{i=1}^{N} a_i c_i = \sum_{i=1}^{N} a_i k_i \qquad (6\text{-}15)$$

and two others in which the common factor a_i inside the sums is replaced, in turn, by b_i and c_i. These equations are the result of taking moments against the coefficients of the unknowns, that is, by multiplying each equation through by its coefficient of $X_\odot$ (namely a_i) and summing over all equations, and so on.

Hence, the final system of equations to be solved, which we obtain by applying the method of least squares to equation (6-12), is

$$X_\odot \sum_{i=1}^{N} \cos^2\delta_i \cos^2\alpha_i + Y_\odot \sum_{i=1}^{N} \cos^2\delta_i \sin\alpha_i \cos\alpha_i + Z_\odot \sum_{i=1}^{N} \sin\delta_i \cos\delta_i \cos\alpha_i = -\sum_{i=1}^{N} \dot{d}_i \cos\delta_i \cos\alpha_i \qquad (6\text{-}16a)$$

$$X_\odot \sum_{i=1}^{N} \cos^2\delta_i \sin\alpha_i \cos\alpha_i + Y_\odot \sum_{i=1}^{N} \cos^2\delta_i \sin^2\alpha_i + Z_\odot \sum_{i=1}^{N} \sin\delta_i \cos\delta_i \sin\alpha_i = -\sum_{i=1}^{N} \dot{d}_i \cos\delta_i \sin\alpha_i \qquad (6\text{-}16b)$$

and

$$X_\odot \sum_{i=1}^{N} \sin\delta_i \cos\delta_i \cos\alpha_i + Y_\odot \sum_{i=1}^{N} \sin\delta_i \cos\delta_i \sin\alpha_i + Z_\odot \sum_{i=1}^{N} \sin^2\delta_i = -\sum_{i=1}^{N} \dot{d}_i \sin\delta_i \qquad (6\text{-}16c)$$

This set of three linear equations can be solved straightaway for $(X_\odot, Y_\odot, Z_\odot)$ because $\dot{d}_i$, α_i, and δ_i are presumed known for each star.

Having obtained its components, we can calculate the speed of the solar motion as

$$S_\odot = (X_\odot^2 + Y_\odot^2 + Z_\odot^2)^{1/2} \qquad (6\text{-}17)$$

and the position of the apex in equatorial coordinates follows from

$$\tan\alpha_A = \frac{Y_\odot}{X_\odot} \qquad (6\text{-}18a)$$

$$\tan\delta_A = \frac{Z_\odot}{(X_\odot^2 + Y_\odot^2)^{1/2}} \qquad (6\text{-}18b)$$

Equation (6-18b) determines δ_A uniquely in the range $-90° \le \delta_A \le 90°$. To resolve the choice of quadrant for α_A, one notes that $\cos\delta_A$ is greater than zero. Hence, the signs of $X_\odot$ and $Y_\odot$ determine the signs of $\cos\alpha_A$ and $\sin\alpha_A$, respectively. Finally, (ℓ_A, b_A), the position of the apex in galactic coordinates, is easily obtained from (α_A, δ_A) by standard formulae of spherical trigonometry. We can then easily compute

$$u_\odot = -S_\odot \cos\ell_A \cos b_A \qquad (6\text{-}19a)$$

$$v_\odot' = S_\odot \sin\ell_A \cos b_A \qquad (6\text{-}19b)$$

$$w_\odot = S_\odot \sin b_A \qquad (6\text{-}19c)$$

Solution from Proper Motions

Consider now equations (6-11b) and (6-11c). We first express $\dot{\alpha} d \cos\delta$ and $\dot{\delta} d$ in terms of practical observational units. Both $\dot{\delta} d$ and $\dot{\alpha} d \cos\delta$ are to be velocities in kilometers per second. Thus if we express d in parsecs as $(1/\pi'')$, where π'' is the parallax in seconds of arc, and use proper motions μ_δ'' and $\mu_\alpha'' \cos\delta$ in seconds of arc per year, then, from equation (3-2), we see that $\dot{\delta} d = (4.74\mu_\delta''/\pi'')$ (km s^{-1}) and $\dot{\alpha} d \cos\delta = (4.74\mu_\alpha'' \cos\delta/\pi'')$ (km s^{-1}). Therefore, equations (6-11b) and (6-11c) can be rewritten as

$$-\dot{x}_i \sin\delta_i \cos\alpha_i - \dot{y}_i \sin\delta_i \sin\alpha_i + \dot{z}_i \cos\delta_i = \frac{4.74\mu_{\delta i}''}{\pi_i''} \qquad (6\text{-}20a)$$

and

$$-\dot{x}_i \sin\alpha_i + \dot{y}_i \cos\alpha_i = \frac{4.74\mu_{\alpha i}'' \cos\delta_i}{\pi_i''} \qquad (6\text{-}20b)$$

Suppose we choose N stars in a region of the sky that is sufficiently small that all stars in it have almost the same α_i and δ_i. Then, by the same line of