ASTR 222 Homework #5

1. Deriving the Expansion History of the Universe (20 points)

Start with the Friedman equation:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2 = -\frac{kc^2}{R^2}$$

First, derive R(t) for a flat, matter-only Universe by integrating the Friedman equation. Then use your expression for R(t) to show that the age of such a Universe is $t_0 = \frac{2}{3}\frac{1}{H_0}$. In doing this, you will find it helpful to remember that the density of matter scales with volume, so that you can rewrite ρ as $\rho_0 R^{-3}$.

Next, in the very early universe (before $t \approx 10,000$ years), the energy density of the photons in the Universe was much higher than the matter density. At these times, we say the universe was "radiation dominated." While the energy density of matter scales as R^{-3} (i.e., its just a volume effect), the energy density of photons scales as R^{-4}, since not only is the volume of the universe going up as it expands, photons are being redshifted and thus have lower energy. This results in an that extra factor of R in the scaling factor for photon energy density.

Use the expression for energy density scaling in place of matter density scaling in the Friedman equation to work out how R(t) scales with time during the radiation dominated era. Compare that expression to what you got in the previous calculation for the flat matter-dominated universe.

Now finally consider a flat Universe with a cosmological constant that drives the expansion. At late times, when the Universe has expanded to very large size, use the Friedman equation to work out how R(t) scales with time in this "Lambda dominated era."

In each of these calculations, make sure to start with the full Friedman equation. Then explain any physical assumptions or justifications for dropping terms from the equation before deriving your solution. Also give good explanations for each step of your derivation; remember, I don't want pages and pages of unexplained math!

2. Cosmological Densities (20 points)

Given our expression for the critical density (the density of matter needed to make the Universe spatially flat by matter alone):

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

Work out the critical density today (at $t = t_0$), in units of $M_\odot/Mpc^3$.

Now work out the density (again in $M_\odot/Mpc^3$) of various objects in the Universe:

- A globular cluster
- The Milky Way galaxy
- The Local Group of galaxies
- The Coma Cluster of galaxies

For any information you need to get (masses, sizes, distances, etc), make sure to cite your sources.

In each case, after calculating the density, calculate an "Omega-object", i.e.:

$$\Omega_{obj} = \frac{\rho_{obj}}{\rho_{crit}}$$

Take some care with units while doing these calculations, and remember that if you measure distances in pc, velocities in km/s, masses in solar masses, and time in Myr, then the gravitational constant can be expressed as $G = 4.43 \times 10^{-3}$ pc $(km/s)^2\ M_\odot^{-1}$. Remember also that $1\,km/s \approx 1\,pc/Myr$.

Also, in each case, explain whether you feel your density estimate of the object is reasonable, or if you think it may be an underestimate or an overestimate. Explain your reasoning.

Finally, do you think any of them are a reasonable estimate of the overall density of the Universe? Again, explain your reasoning, and explain physically, not just based on a comparison of Ω_{obj} to some expected value of Ω_m.

3. Observational Cosmology (20 points)

(Note: Detailed calculations of apparent magnitudes, sizes, "distances", times, etc as a function of redshift under different cosmologies can be very complicated mathematically, so rather than working things out analytically, we are going to use astropy's cosmology package to do things for us. Instructions for using this package are given at the end of the assignment.)

In the local universe, the massive elliptical galaxy M49 lies at a distance of 16.5 Mpc, has a total luminosity of $L_{V,tot} = 1.6 \times 10^{11}\ L_\odot$ and a physical half-light radius of $r_e = 25$ kpc. Using our standard (i.e., non-cosmological) methods for calculating apparent magnitudes, angular sizes, etc, calculate M49's total apparent V magnitude, angular half-light radius (in arcsec), and effective surface brightness (in magnitudes per square arcsecond). Remember that the effective surface brightness measures the average surface brightness inside the half-light radius, and we can measure that from $\langle\mu\rangle_e = m_{1/2} + 2.5 \log A$, where $m_{1/2}$ is the apparent magnitude of half the total light, and A is the area (in square arcseconds) inside the half-light radius.

Now imagine observing M49 if it were redshifted by different amounts, and we'll ask how the observed properties of the galaxy change as we move it to higher and higher redshift. We'll compare the properties calculated using a simple Hubble law (which will be incorrect at higher redshift) vs those calculated in a cosmologically correct way. The idea here will be to find out how fast we get bad answers by doing things the simple (and cosmologically incorrect) way.

For the first three rows, given the redshift calculate the distance, apparent magnitude, and angular half-light radius of M49 at each redshift simply using a simple Hubble's law ($v = H_0 d$) and the regular expressions for apparent magnitude and angular size. Use $H_0 = 72$ km/s/Mpc for this. ***Remember, we know doing the calculation this way is wrong at higher redshift, but what we are going to look at is how quickly does it get wrong.***

Now let's do the calculation the right way. Adopt our "best" cosmological model ($H_0 = 72$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$) and use astropy's cosmology package to work out the rest of the entries in the table below, which will give the "cosmologically correct" observed properties of M49 at different redshifts.

Property	$z = 0.00396$	$z = 0.01$	$z = 0.1$	$z = 0.5$	$z = 1$	$z = 5$
Values Using a simple Hubble Law distance						
Hubble Law Distance (Mpc)						
Hubble Distance Apparent V mag						
Hubble Distance Half-light radius (arcsec)						
Cosmologically correct calculations						
Luminosity Distance (Mpc)						
Apparent V mag						
Angular Diameter Distance (Mpc)						
Half-light radius (arcsec)						
Effective surface brightness (mags per sq. arcsec)						
Age of Universe at observed redshift (Gyr)						
Look back time to observed redshift (Gyr)						

Note that since $z = 0.00396$ is the redshift that corresponds to a distance of 16.5 Mpc, the "cosmologically correct" numbers at this redshift should pretty much match what you calculated by hand, so that would be a good check to make sure you're doing everything correctly.

After completing the table, answer the following questions:

- If you are used to using plain old Hubble Law distances, how far out in redshift can you go before your "Hubble law magnitudes" are wrong by 0.1 mag or more? How far out can you go before your angular half-light radius will be wrong by 10% or more?

- Now look at the cosmologically correct half light radius as you go to higher redshift. Assume that you can accurately measure half-light radii as long as they are bigger than 0.5 arcsec. How far out in redshift can you go before the galaxy becomes too small to accurately measure its half-light radius?

- In class, when talking about galaxies, we said that one of the reasons that surface brightness is so important is that it is "distant invariant" – that is, if you imagine moving a galaxy to larger distances, its observed surface brightness doesn't change. Look at the cosmologically-correct surface brightness calculation: is it also "redshift invariant"? Or does M49 show a brighter or fainter surface brightness as you move it to higher redshift? Look at how apparent magnitude and angular size change with redshift and explain why the observed surface brightness changes the way it does.

- The apparent V magnitude (and surface brightness) you calculate this way don't account for another important effect: the fact that when we look at objects at different redshift using single filter (say, the V-band filter), the light we are seeing actually was emitted by the galaxy at much bluer wavelengths, before being redshifted to the wavelengths we observe. If the V filter has a central wavelength of $\lambda_c = 5500$ Å, what wavelength does that correspond to in the **emitted** spectrum of M49 at each of those redshifts in the table above? If you were to actually correct for this effect (a complex process called a "K-correction"), would that make the observed magnitudes and surface brightnesses brighter, fainter, or would they not change at all? To answer this, think about what the emitted spectrum from a galaxy like M49 ought to look like, and make sure to explain your reasoning.

- Finally, if we want to be looking back to times when the universe was younger than 2 Gyr, how far out in redshift would we have to look? What about if we only wanted to look back to times earlier half the Universe's current age?

Using Astropy's Cosmology package

```python
from astropy import cosmology
from astropy import units as u

# set up the universe with the parameters you want:
universe=cosmology.LambdaCDM(H0=72, Om0=0.3, Ode0=0.7)

# set redshift you want to explore
z=0.3

# luminosity distance (DL) is what you use when calculating apparent magnitudes,
# i.e., m-M = 5*np.log10(DL)+5 (and remember DL should have units of parsecs!)
DL=universe.luminosity_distance(z)
print('Luminosity distance at z={} is {:.2f}'.format(z,DL))
# but before you stuff it into the mag-distance equation, make sure DL is in parsecs,
# and that you've converted it to a regular python number:
DL_pc = DL.to(u.pc).value
# and now you can calculate things using DL_pc

# angular diameter distance (DA) is what you use when converting between
#angular size (alpha) and physical size (d_phys):
# d_phys = alpha[arcsec]*DA/206265
DA=universe.angular_diameter_distance(z)
print('Angular diameter distance at z={} is {:.2f}'.format(z,DA))
# again, you can convert it to a regular number and unit, for example kpc
DA_kpc = DA.to(u.kpc).value
# and now you can calculate things using DA_kpc

# ages and lookback times you can get this way:
print('Universe age at z={} is {:.2f}'.format(z,universe.age(z)))
print('Lookback time to z={} is {:.2f}'.format(z,universe.lookback_time(z)))
```